GEOMETRIC INEQUALITIES AND RIGIDITY THEOREMS ON EQUATORIAL SPHERES

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ABSTRACT. We prove rigidity for hypersurfaces in the unit $(n+1)$-sphere whose scalar curvature is bounded below by $n(n-1)$, without imposing constant scalar curvature nor constant mean curvature. The lower bound $n(n-1)$ is critical in the sense that some natural differential operators associated to the scalar curvature may be fully degenerate at geodesic points and cease to be elliptic. We overcome the difficulty by developing an approach to investigate the geometry of level sets of a height function, via new geometric inequalities.

1. INTRODUCTION

Hypersurfaces in $\mathbb{S}^{n+1}$ of either *constant scalar curvature* or *constant mean curvature* (including minimal hypersurfaces) have been extensively studied in the literature. For example, for constant scalar curvature, the pioneering work of Cheng–Yau [5] says that if a hypersurface M in $\mathbb{S}^{n+1}$ has constant scalar curvature which is greater than or equal to $n(n-1)$ and non-negative sectional curvature, then M is either an n-sphere or a Riemannian product of a p-sphere and an $(n-p)$-sphere. They introduced a self-adjoint operator [5] which has been used by many other people to study hypersurfaces of constant scalar curvature, under various additional conditions (see, for example, [1, 12]).

In this paper, we consider hypersurfaces M in $\mathbb{S}^{n+1}$ whose scalar curvature R is bounded below by $n(n-1)$, namely, $R \geq n(n-1)$. We do *not* assume constant scalar curvature nor constant mean curvature. We would like to prove the rigidity of M under certain boundary condition. First, the boundary condition is necessary to derive rigidity results, because small spheres and their slight perturbation have sufficiently large scalar curvature. Furthermore, the lower bound $n(n-1)$ is critical because, by the Gauss equation, $R \geq n(n-1)$ implies that $H^2 \geq |A|^2$. Thus, the mean curvature H may change signs locally through the *geodesic* points, i.e., the points where the second fundamental form A vanishes. From the analytic aspect, the geometric operators, such as Cheng–Yau's operator, the linearized scalar curvature operator, and the scalar curvature flow, are no longer globally elliptic nor parabolic. Also, these operators are totally degenerate at the geodesic points. Therefore, the theory of maximum principle cannot be applied directly. (We remark that, if one assumes that M is contained in the hemisphere and that M has constant scalar curvature, then the desired

ellipticity automatically holds; and hence, M is a sphere by applying the Alexandrov reflection principle (see Korevaar [11]).) Rather than sticking to these geometric operators, we develop a different approach using some new geometric inequalities, motivated by our recent work [10].

Another important motivation of this paper comes from Min–Oo's conjecture. Let M be an n-dimensional compact Riemannian manifold of scalar curvature $R \geq n(n-1)$ with boundary ∂M. Suppose that ∂M is isometric to the unit sphere $\mathbb{S}^{n-1}$ and is totally geodesic in M. The conjecture states that M is isometric to the hemisphere $\mathbb{S}^n_+$. There have been many interesting results related to the conjecture. For example, Hang–Wang [7, 8] proved the conjecture under the condition that, either g is conformal to the standard sphere metric, or the Ricci curvature satisfies $\mathrm{Ric} \geq (n-1)g$. By assuming positive Ricci curvature on M and an isoperimetric condition on ∂M, Eichmair [6] proved the conjecture for $n = 3$. Recently, Brendle–Marques–Neves [3] constructed counter-examples to Min–Oo's conjecture by using beautiful deformation techniques. For other recent results, we refer to the excellent survey by Brendle [2] and the references therein.

In an early work [9], we proved Min–Oo's conjecture for M being a hypersurface with boundary in either Euclidean space or hyperbolic space, by applying the strong maximum principle to the mean curvature operator. It is a natural continuation to study the case when M is in $\mathbb{S}^{n+1}$. However, the method in [9] does not apply to the spherical case, mainly due to the failure of ellipticity. Nevertheless, we are able to overcome the difficulty and obtain the following result.

Theorem 1. *Let M be a connected and embedded C^{n+1} hypersurface in $\mathbb{S}^{n+1}$ with boundary ∂M. Suppose that $(M, \partial M)$ satisfies the following conditions:*

1. *The scalar curvature of M satisfies $R \geq n(n-1)$;*
2. *M is tangent to a great n-sphere $\mathbb{S}^n$ at ∂M, and ∂M is the great $(n-1)$-sphere $\mathbb{S}^{n-1}$.*

Then, M is the hemisphere $\mathbb{S}^n_+$.

Theorem 1 confirms Min–Oo's conjecture when M is a hypersurface in $\mathbb{S}^{n+1}$, under a weaker boundary condition. In fact, we prove the following stronger version, where ∂M is not assumed to be the great $(n-1)$-sphere (see Definition 4.5). We remark that the boundary condition below is necessary, in view of Example 6.2.

Theorem 2. *Let M be a connected and embedded C^{n+1} hypersurface in $\mathbb{S}^{n+1}$ with boundary ∂M. Suppose that $(M, \partial M)$ satisfies the following conditions:*

1. *The scalar curvature of M satisfies $R \geq n(n-1)$;*
2. *M is tangent to a great n-hemisphere $\mathbb{S}^n_+$ at ∂M.*

Then, M is a portion of the hemisphere $\mathbb{S}^n_+$.

As a direct corollary, for *closed* (i.e, compact without boundary) hypersurfaces in $\mathbb{S}^{n+1}$, we have the following result.

Theorem 3. *Let M be a closed, connected, and embedded C^{n+1} hypersurface in $\mathbb{S}^{n+1}$ satisfying $R \geq n(n-1)$. If M is tangent to a great n-sphere at a great $(n-1)$-sphere. Then, M is the great n-sphere; namely, M is an equatorial sphere in $\mathbb{S}^{n+1}$.*

We introduce a geometric inequality which quantifies geometry of the level sets in M of a height function. Although the maximum principle type argument fails to work on M, we are able to apply the maximum principle to the level sets. This relies on the geometric inequality which relates the scalar curvature and mean curvature of M in $\mathbb{S}^{n+1}$ to the mean curvature of the level sets of a height function. The analogous situation occurs for hypersurfaces in Euclidean space with nonnegative scalar curvature. By the new technique, we have shown that if M is closed hypersurface in $\mathbb{R}^{n+1}$ with nonnegative scalar curvature, then it must be mean convex [10]. However, in the spherical case the geometric inequality is more involved. We deform a family of conic hypersurfaces to obtain the desired level set in M, where the inequality is utilized at full strength. We believe that this approach would have further applications in more general ambient manifolds. In the following theorem, we derive the geometric inequality for a large class of ambient spaces.

Let (N, g) be an n-dimensional Riemannian manifold, M a C^2 hypersurface in the product manifold $(N \times \mathbb{R}, g + dt^2)$, and ν the unit normal vector field to M in $(N \times \mathbb{R}, g + dt^2)$. Let $\Sigma = \{t = \epsilon\} \cap M$ be a regular level hypersurface in M, and η the unit normal to Σ in $(N \times \{\epsilon\}, g)$.

Theorem 4. *Let M and Σ be given as above. Let $N \times \mathbb{R}$ endow with a metric $\phi^{-2}(g + dt^2)$, for a positive function $\phi \in C^1(N \times \mathbb{R})$. We denote by $\bar{A}$ and $\bar{A}_\Sigma$ the shape operators of M in $(N \times \mathbb{R}, \phi^{-2}(g + dt^2))$ and Σ in $(N \times \{\epsilon\}, \phi^{-2}(\cdot, \epsilon)g)$, respectively; and denote by $\bar{H}$, $\bar{H}_\Sigma$ the corresponding mean curvatures. Then, at any point of Σ,*

$$\begin{aligned}&\bar{H}\left[\langle \nu, \eta\rangle \bar{H}_\Sigma + (n-1)\langle \nu, \partial_t\rangle \phi_t\right]\\ &\geq \frac{1}{2}\left(\bar{H}^2 - |\bar{A}|^2\right) + \frac{n}{2(n-1)}\left[\langle \nu, \eta\rangle \bar{H}_\Sigma + (n-1)\langle \nu, \partial_t\rangle \phi_t\right]^2,\end{aligned} \tag{1.1}$$

where $\phi_t = \partial\phi/\partial t$, and $\langle \cdot, \cdot\rangle$ is with respect to the product metric $g + dt^2$. The equality in (1.1) holds at a point of Σ if and only if M and Σ satisfy the following conditions at the point:

(i) *Σ is umbilic in $(N \times \{\epsilon\}, \phi^{-2}(\cdot, \epsilon)g)$. Let κ be the principal curvature.*
(ii) *$M \subset (N \times \mathbb{R}, \phi^{-2}g)$ has principal curvature $\langle \nu, \eta\rangle\kappa + \langle \nu, \partial_t\rangle \phi_t$ with multiplicity at least $n - 1$.*

As an independent result, we obtain below a rigidity theorem for hypersurfaces M in $\mathbb{R}^{n+1}$ with nonnegative scalar curvature, and the boundary ∂M being *planar*, i.e., ∂M is contained in a hyperplane and M is tangent to the hyperplane at ∂M from the region enclosed by ∂M (see also Definition 5.2). The boundary condition here is necessary, as shown in Example 6.1..

Theorem 5. *Let M be a compact, connected, and embedded C^{n+1} hypersurface in $\mathbb{R}^{n+1}$ with boundary ∂M. Suppose that ∂M is planar. If the scalar curvature of M is nonnegative, then M is contained in a hyperplane; namely, M is Euclidean flat.*

The rest of this paper is organized as follows. The geometric inequalities are derived in Section 2 and Section 3. The rigidity results for hypersurfaces in spheres are proved in Section 4, while the result in Euclidean space is proved in Section 5. In Section 6, we provide two examples to indicate that the boundary conditions imposed in Theorem 2 and Theorem 5 are indispensable.

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2. Geometric inequalities on Product manifolds

Let (N, g) be a n-dimensional complete Riemannian manifold, and ∇ the Levi–Civita connection of g. Consider the product manifold $N \times \mathbb{R}$ endowed with the product metric $g + dt^2$, which is also denoted by $\langle \cdot, \cdot \rangle$. Since $(\mathbb{R}, dt^2)$ is flat, by abuse of notation, we also use ∇ to denote the Levi-Civita connection on the product manifold. Let f be a C^2 function over an open subset $\Omega \subset N$ and $\{x^1, \dots, x^n\}$ a coordinate system in a neighborhood of $x \in \Omega$. We denote

$$\partial_i = \frac{\partial}{\partial x^i}, \quad f_i = \frac{\partial f}{\partial x^i}, \quad f^i = \sum_{k=1}^n g^{ik} f_k,$$
$$f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}, \quad \nabla_i = \nabla_{\partial_i}, \quad \text{for all } i, j = 1, \dots, n,$$

and $\partial_{n+1} = \partial/\partial t$. Then, for $i = 1, \dots, n$, $e_i = \partial_i + f_i \partial_{n+1}$ is tangent to the graph of f, and $\{e_1, \dots, e_n\}$ forms a coordinate frame. Let ν be the unit normal vector of the graph of f in $N \times \mathbb{R}$. The shape operator A^i_j with respect to ν is defined by $A^i_j = \sum_{k=1}^n g^{ik} \langle \nu, \nabla_{e_k} e_j \rangle$, for $i, j = 1, \dots, n$. By a direct calculation, the shape operator with respect to the upward unit normal vector is given by

$$\begin{aligned} A^i_j &= \sum_{k=1}^n \left(g^{ik} - \frac{f^i f^k}{1 + |\nabla f|^2} \right) \frac{\nabla_j \nabla_k f}{\sqrt{1 + |\nabla f|^2}} \\ &= \nabla_j \left(\frac{f^i}{\sqrt{1 + |\nabla f|^2}} \right). \end{aligned} \tag{2.1}$$

Let M be a C^2 hypersurface in the product manifold $(N \times \mathbb{R}, g + dt^2)$ and ν the unit normal to M in $N \times \mathbb{R}$. Consider the (nonempty) level set $\Sigma = M \cap \{t = \epsilon\}$. By the Morse–Sard theorem (see [16] for example), for almost every ϵ, Σ is a C^2 submanifold of M and also a C^2 submanifold in $N \times \{\epsilon\}$. Denote by η the unit normal vector to Σ in $(N \times \{\epsilon\}, g)$ and by H_Σ the mean curvature of Σ with respect to η. Also, we denote by $\sigma_1(A)$ the trace of a matrix A.

Lemma 2.1. *Let $\Sigma = M \cap \{t = \epsilon\}$ for some regular value ϵ. For a point $p \in \Sigma \subset M$, let A be the shape operator of M in $(N \times \mathbb{R}, g + dt^2)$ with respect to the unit normal ν, and A_Σ the shape operator of Σ in $(N \times \{\epsilon\}, g)$ with respect to η. Then, there exists a coordinate chart $(x^1, \dots, x^n)$ of p in M such that $(x^2, \dots, x^n)|_\Sigma$ is a coordinate chart of p in Σ, and that*

$$(A|1) = \langle \nu, \eta \rangle A_\Sigma \qquad \text{at } p,$$

where $(A|1)$ stands for the $(n-1) \times (n-1)$ minor in the matrix A with the first row and first column deleted, and $\langle \cdot, \cdot \rangle$ is with respect to $g + dt^2$. In particular, taking the trace of the above identity, we have

$$\sigma_1(A|1) = \langle \nu, \eta \rangle H_\Sigma \quad \text{at } p.$$

Proof. Without loss of generality, we can assume that $\langle \nu, \eta \rangle \geq 0$ at p, for otherwise we can just replace η by $-\eta$. We divide the proof into the following two cases:

Case 1: Suppose that $\langle \nu, \eta \rangle < 1$ at p. Then, M, locally near $p = (x, \epsilon)$, is the graph of f over a domain $\Omega \subset N$. Let ν be the upward unit normal to M given by

$$\nu = \frac{1}{\sqrt{1 + |\nabla f|^2}} \left(-\sum_{i=1}^n f^i \partial_i + \partial_{n+1} \right).$$

We can choose a coordinate system $\{x^1, \dots, x^n\}$ near x so that at x, $g_{ij} = \delta_{ij}$, $\partial g_{ij}/\partial x^k = 0$, and

$$f_1 = |\nabla f|, \quad f_\alpha = 0 \text{ for } \alpha = 2, \dots, n.$$

By (2.1), we obtain at p that

$$A^\alpha_\beta = \frac{f_{\alpha\beta}}{\sqrt{1 + |\nabla f|^2}} \qquad \text{for all } 2 \leq \alpha, \beta \leq n.$$

On the other hand, because Σ near p is given by $\{x \in \Omega : f(x) = \epsilon\}$ and $\langle \nu, \eta \rangle \geq 0$ at p, we have $\eta = -\sum_{k=1}^n f^k \partial_k / |\nabla f|$. Then, at p,

$$(A_\Sigma)^\alpha_\beta = \nabla_\beta \left(\frac{f^\alpha}{|\nabla f|} \right) = \frac{f_{\alpha\beta}}{|\nabla f|} \qquad \text{for all } 2 \leq \alpha, \beta \leq n.$$

Since $\langle \nu, \eta \rangle = |\nabla f| / \sqrt{1 + |\nabla f|^2}$, we obtain

$$(A|1) = \langle \nu, \eta \rangle A_\Sigma \quad \text{at } p.$$

Case 2: Suppose that $\langle \nu, \eta \rangle = 1$ at p. That is, $\nu = \eta$ at p. Let $\{e_1, \dots, e_n, e_{n+1}\}$ be a local frame in a neighborhood of p in $N \times \mathbb{R}$ such that $e_{n+1} = \nu$, and that $\{e_1, \dots, e_n\}$ is a coordinate frame on M where $e_1|_\Sigma = \partial_t$, $\{e_2, \dots, e_n\}$ restricted to a coordinate frame on Σ. Then, the shape operator of M in $N \times \mathbb{R}$ at p is $A^i_j = \sum_{k=1}^n g^{ik} g(\nu, \nabla_{e_k} e_j)$ for $1 \le i, j \le n$. In particular, for $\alpha, \beta = 2, \dots, n$,

$$A^\alpha_\beta(p) = \sum_{\gamma=2}^n g^{\alpha\gamma} g(\nu, \nabla_{e_\gamma} e_\beta).$$

Observe that the shape operator of Σ in $(N \times \{\epsilon\}, g)$ at p is

$$(A_\Sigma)^\alpha_\beta = \sum_{\gamma=2}^n g^{\alpha\gamma} g(\eta, \nabla_{e_\gamma} e_\beta) = A^\alpha_\beta \quad \text{for all } \alpha, \beta = 2, \dots, n,$$

since $\nu = \eta$ at p. We finish the proof for Case 2. □

For a C^2 hypersurface M in $(N \times \mathbb{R}, g + dt^2)$, the proposition below follows from the Gauss equation.

Proposition 2.2. *Let M be a hypersurface in $(N \times \mathbb{R}, g + dt^2)$ with the induced metric g^M. Let ν be the unit normal vector to M. Denote by A the shape operator and by H the mean curvature with respect to ν, respectively. Then,*

$$H^2 - |A|^2 = R(g^M) - R(g) + 2\mathrm{Ric}_g(\nu', \nu'),$$

where $\nu' = \nu - \langle \nu, \partial_t \rangle \partial_t$ in which $\langle \cdot, \cdot \rangle$ is with respect to $g + dt^2$, and R stands for the scalar curvature.

Proof. Applying the Gauss equation to M in $N \times \mathbb{R}$ yields

$$R(g + dt^2) = 2\mathrm{Ric}_{g+dt^2}(\nu, \nu) + R(g^M) - H^2 + |A|^2.$$

By the curvature formulas of a product metric (see, for example, [14, p. 89]), we have

$$R(g + dt^2) = R(g), \quad \mathrm{Ric}_{g+dt^2}(\nu, \nu) = \mathrm{Ric}_g(\nu', \nu').$$

□

Now we are able to prove the following geometric equality for product metrics:

Theorem 2.3. *Let (N, g) be an n-dimensional Riemannian manifold, M be a C^2 hypersurface in the product manifold $(N \times \mathbb{R}, g + dt^2)$, ν the unit normal to M in $N \times \mathbb{R}$, and H, $R(g^M)$ the mean curvature and scalar curvature of M in $N \times \mathbb{R}$, respectively. Let $\Sigma = M \cap \{t = \epsilon\}$ for some regular value ϵ, and η the unit normal to Σ in $N \times \{\epsilon\}$. Then, at any point of Σ, we have*

$$\begin{aligned} \langle \nu, \eta \rangle H H_\Sigma \ge \frac{1}{2} R(g^M) - \frac{1}{2} R(g) + \langle \nu, \eta \rangle^2 \mathrm{Ric}_g(\eta, \eta) \\ + \frac{n}{2(n-1)} \langle \nu, \eta \rangle^2 H_\Sigma^2, \end{aligned} \tag{2.2}$$

in which $\langle\cdot,\cdot\rangle$ is with respect to the metric $g+dt^2$, H_Σ is the mean curvature of Σ in $N\times\{\epsilon\}$, and $R(g)$ and Ric_g are the scalar curvature and Ricci curvature of (N,g), respectively. The equality in (2.2) holds at a point of Σ if and only if M and Σ satisfy the following conditions at the point:

(i) *Σ is umbilic in $N\times\{\epsilon\}$. Denote by κ the principal curvature.*
(ii) *$M\subset N\times\mathbb{R}$ has principal curvature $\langle\nu,\eta\rangle\kappa$ with multiplicity at least $n-1$.*

Proof. The inequality (2.2) follows immediately from applying the linear algebraic identity ([10], see also Proposition 2.4 below) to Lemma 2.1 with $2\sigma_2(A)=H^2-|A|^2$ and Proposition 2.2 with $\nu'=\langle\nu,\eta\rangle\eta$. $\square$

Proposition 2.4 ([10])**.** *Let $A=(a_{ij})$ be a real $n\times n$ matrix with $n\geq 2$. Denote*

$$\sigma_1(A)=\sum_{i=1}^n a_{ii},\quad \sigma_1(A|1)=\sum_{i=2}^n a_{ii},\quad \sigma_2(A)=\sum_{1\leq i<j\leq n}(a_{ii}a_{jj}-a_{ij}a_{ji}).$$

Then, we have

$$\begin{aligned}\sigma_1(A)\sigma_1(A|1)=\sigma_2(A)+\frac{n}{2(n-1)}[\sigma_1(A|1)]^2+\sum_{1\leq i<j\leq n}a_{ij}a_{ji}\\ +\frac{1}{2(n-1)}\sum_{2\leq i<j\leq n}(a_{ii}-a_{jj})^2.\end{aligned}\tag{2.3}$$

In particular, if $a_{ij}a_{ji}\geq 0$ for all $1\leq i<j\leq n$, then

$$\sigma_1(A)\sigma_1(A|1)\geq\sigma_2(A)+\frac{n}{2(n-1)}[\sigma_1(A|1)]^2,$$

where "$=$" holds if and only if $a_{22}=\cdots=a_{nn}$ and $a_{ij}a_{ji}=0$ for all $1\leq i<j\leq n$.

3. Geometric inequalities for conformal metrics

In this section, we shall generalize the geometric inequality to the conformal product metrics. Namely, we would like to prove Theorem 4. Let us first recall a general formula for the shape operator under conformal transformation (see, for example, [4, p. 183]).

Proposition 3.1. *Let $(\mathcal{N},g)$ be an m-dimensional manifold, $\mathcal{M}$ a hypersurface in $\mathcal{N}$, and μ the unit normal vector to $\mathcal{M}$ with respect to g. Let $\{e_2,\dots,e_m\}$ be a local frame field in $\mathcal{M}$, and A^i_j the shape operator of $\mathcal{M}$ with respect to g, μ, and $\{e_2,\dots,e_n\}$. Consider $\bar g=\phi^{-2}g$ for some positive function ϕ in $\mathcal{N}$. Let $\bar A^i_j$ be the shape operator on $\mathcal{M}$ with respect to $\bar g$, $\phi\mu$, and $\{e_2,\dots,e_n\}$. Then,*

$$\bar A^i_j=\phi A^i_j+\mu(\phi)\delta^i_j,\qquad 2\leq i,j\leq m.$$

As a consequence,

$$\bar H=\phi H+(m-1)\mu(\phi)\tag{3.1}$$

where H and $\bar{H}$ denote the mean curvatures of M with respect to g and $\bar{g}$, respectively.

Let us recall the setting in Theorem 4: Let (N, g) be an n-dimensional Riemannian manifold, M a C^2 hypersurface in the product manifold $(N \times \mathbb{R}, g + dt^2)$, and ν the unit normal vector field to M in $N \times \mathbb{R}$. Let $\Sigma = \{t = \epsilon\} \cap M$ be a regular level hypersurface in M, and η the unit normal to Σ in $(N \times \{\epsilon\}, g)$. We denote by A and A_Σ the shape operators of $M \subset N \times \mathbb{R}$ and $N \times \{\epsilon\}$ with respect to ν and η, respectively. Then, $\sigma_1(A) = H$ and $\sigma_1(A_\Sigma) = H_\Sigma$. Let $N \times \mathbb{R}$ endow with a metric $\phi^{-2}(g + dt^2)$, for a positive function $\phi \in C^1(N \times \mathbb{R})$. We denote by $\bar{A}$ and $\bar{A}_\Sigma$ the shape operators of M in $(N \times \mathbb{R}, \phi^{-2}(g + dt^2))$ and Σ in $(N \times \{\epsilon\}, \phi^{-2}(\cdot, \epsilon)g)$, respectively; and denote by $\bar{H}$, $\bar{H}_\Sigma$ the corresponding mean curvatures.

To prove Theorem 4, it suffices to establish the following lemma, similar to Lemma 2.1. Then, Theorem 4 follows immediately by substituting (3.2) into Proposition 2.4.

Lemma 3.2. *With the notation in Theorem 4, for any point $p \in \Sigma \subset M$, there exists a coordinate chart $(x^1, \dots, x^n)$ of p in M such that $(x^2, \dots, x^n)$ is restricted to be a coordinate chart of p in Σ, and that*

$$(\bar{A}|1) = \langle \nu, \eta \rangle \bar{A}_\Sigma + \langle \nu, \partial_t \rangle \phi_t I_{n-1} \qquad \text{at } p,$$

where $\langle \cdot, \cdot \rangle$ is with respect to $g + dt^2$, $(\bar{A}|1)$ is the matrix obtained by deleting the first row and first column of $\bar{A}$, and I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. In particular,

$$\sigma_1(\bar{A}|1) = \langle \nu, \eta \rangle \bar{H}_\Sigma + (n-1)\langle \nu, \partial_t \rangle \phi_t \qquad \text{at } p. \tag{3.2}$$

Proof. By Lemma 2.1 there exists a coordinate chart $(x^1, \dots, x^n)$ near p in M, such that $(x^2, \dots, x^n)$ is restricted to be a coordinate chart of p in Σ, and that

$$(A|1) = \langle \nu, \eta \rangle A_\Sigma. \tag{3.3}$$

Fix this coordinate chart. Applying Lemma 3.1 to $(x^2, \dots, x^n)|_\Sigma$ yields that

$$\bar{A}_\Sigma = \phi A_\Sigma + \eta(\phi) I_{n-1}. \tag{3.4}$$

By applying Lemma 3.1 again to $(x^1, \dots, x^n)$, we obtain

$$\begin{aligned}
(\bar{A}|1) &= \phi(A|1) + \nu(\phi) I_{n-1} \\
&= \langle \nu, \eta \rangle \phi A_\Sigma + \nu(\phi) I_{n-1}, \qquad \Big(\text{by } (3.3)\Big) \\
&= \langle \nu, \eta \rangle \bar{A}_\Sigma + [\nu(\phi) - \langle \nu, \eta \rangle \eta(\phi)] I_{n-1}, \qquad \Big(\text{by } (3.4)\Big) \\
&= \langle \nu, \eta \rangle \bar{A}_\Sigma + \langle \nu, \partial_t \rangle \phi_t I_{n-1}.
\end{aligned}$$

□

As corollaries of Theorem 4, we obtain the following results for hypersurfaces in Euclidean space $(\mathbb{R}^{n+1}, g_0)$, and hypersurfaces in a unit sphere

$(\mathbb{R}^{n+1}, g_S)$, where

$$g_S = \phi^{-2} g_0, \qquad \text{and} \quad \phi = \frac{1 + \sum_{i=1}^{n+1} (x^i)^2}{2}.$$

Corollary 3.3. *Let M be a hypersurface in $(\mathbb{R}^{n+1}, g_0)$. Denote by Σ the regular level set of a height function, say $\Sigma = M \cap \{x^{n+1} = \epsilon\}$ for a regular value ϵ. Let ν and η be the unit normal vectors to M in $(\mathbb{R}^{n+1}, g_0)$ and Σ in $(\mathbb{R}^n \times \{\epsilon\}, g_0)$, respectively. Then*

$$g_0(\nu, \eta) H H_\Sigma \geq \frac{R}{2} + \frac{n}{2(n-1)} g_0(\nu, \eta)^2 H_\Sigma^2,$$

where the equality holds at a point of Σ if and only if (M, Σ) satisfies the following conditions at the point:

(i) *Σ is umbilic in $(\mathbb{R}^n \times \{\epsilon\}, g_0)$. Denote by κ the principal curvature.*
(ii) *M has principal curvature $g_0(\nu, \eta)\kappa$ in $(\mathbb{R}^{n+1}, g_0)$ with multiplicity at least $n-1$.*

Corollary 3.3 recovers a result in our previous work [10, Theorem 2.2], which is employed to prove Lemma 5.1. The following corollary will be used to prove Lemma 4.3.

Corollary 3.4. *Let M be a hypersurface in $(\mathbb{R}^{n+1}, g_S)$. Denote by Σ the regular level set of a height function, say $\Sigma = M \cap \{x^{n+1} = \epsilon\}$ for a regular value ϵ. Let ν and η be the unit normal vectors to M in $(\mathbb{R}^{n+1}, g_0)$ and Σ in $(\mathbb{R}^n \times \{\epsilon\}, g_0)$, respectively. Then*

$$\begin{aligned}
&\bar{H} \left[g_0(\nu, \eta) \bar{H}_\Sigma + (n-1) g_0(\nu, \partial_{n+1}) x^{n+1} \right] \\
&\geq \frac{1}{2}[R - n(n-1)] + \frac{n}{2(n-1)} \left[g_0(\nu, \eta) \bar{H}_\Sigma + (n-1) g_0(\nu, \partial_{n+1}) x^{n+1} \right]^2,
\end{aligned}$$

where the equality holds at a point of Σ if and only if (M, Σ) satisfies the following conditions at the point:

(i) *Σ is umbilic in $(\mathbb{R}^n \times \{\epsilon\}, g_S)$. Denote by κ the principal curvature.*
(ii) *M has principal curvature $g_0(\nu, \eta)\kappa + g_0(\nu, \partial_{n+1})\phi_{n+1}$ in $(\mathbb{R}^{n+1}, g_S)$ with multiplicity at least $n-1$.*

4. Rigidity in spheres

In this section, we study a compact hypersurface M in $\mathbb{S}^{n+1}$ satisfying $R \geq n(n-1)$ with nonempty boundary ∂M. Here we denote by $\mathbb{S}^k$ the k-dimensional unit sphere, and by $\mathbb{S}^k_+$ the *closed* k-dimensional hemisphere. We will make use of the following quasi-linear elliptic operator $H(u)$.

Definition 4.1. Let W be an open subset in $\mathbb{R}^n$ and $u \in C^2(W)$. Define the operator

$$H(u) = \frac{1+|x|^2+u^2}{2} \sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2} \right) \frac{u_{ij}}{\sqrt{1+|Du|^2}} + \frac{n}{\sqrt{1+|Du|^2}} (u - x \cdot Du).$$

By (3.1), $H(u)$ is the mean curvature of the graph of u in $(\mathbb{R}^{n+1}, g_S)$ with respect to the upward unit normal vector

$$\nu = \phi \frac{(-Du, 1)}{\sqrt{1+|Du|^2}}.$$

Here $(g_S)_{ij} = \phi^{-2}\delta_{ij}$ and $\phi = [1 + \sum_{i=1}^{n+1} (x^i)^2]/2$.

Proposition 4.2. *Let W be an open subset in $\mathbb{R}^n$, $p \in \partial W$. Denote by $B(p)$ an open ball centered at p in $\mathbb{R}^n$. Let $u \in C^2(\overline{B(p) \cap W})$ satisfy that $u = 0, |Du| = 0$ on $B(p) \cap \partial W$. If $H(u) > 0$ on $B(p) \cap W$, then*

$$\{x \in B(p) \cap W : u(x) > 0\} \neq \emptyset.$$

Proof. Suppose on the contrary $u \leq 0$ on $B(p) \cap W$. If $u = 0$ somewhere in $B(p) \cap W$, then u attains a local maximum, which contradicts $H(u) > 0$. Thus, u is strictly negative on $B(p) \cap W$. Furthermore, because $B(p) \cap W$ is open, there is a smaller open ball B contained in $B(p) \cap W$ so that ∂B touches $q \in B(p) \cap \partial W$. Namely, ∂W satisfies an interior sphere condition at q. Because $u < 0$ and $H(u) > 0$ on B and $u(q) = 0$, we conclude that $|Du|(q) \neq 0$, by the Hopf boundary point lemma (see, for example, [9, p. 359]). This leads a contradiction. □

Denote by $B_a \subset \mathbb{R}^n$ the open ball of radius a centered at the origin, and denote $S_a = \partial B_a$.

Lemma 4.3. *Let $W \subset B_1$ be an open set in $\mathbb{R}^n$. Suppose that $W \cap (B_1 \setminus B_a) \neq \emptyset$ for some $0 < a < 1$. Let $u \in C^2(\overline{W \cap (B_1 \setminus B_a)})$ satisfy that $u = 0, |Du| = 0$ on $\partial W \cap (B_1 \setminus B_a)$. If the scalar curvature R of the graph of u in $(\mathbb{R}^{n+1}, g_S)$ satisfies $R \geq n(n-1)$, then $H(u) = 0$ at some points in $W \cap (B_1 \setminus B_a)$. As a consequence, $H(u) = 0$ at a sequence of points in W which converges to some point in ∂W.*

Proof. Suppose, on the contrary, that $H(u)$ is never zero over $W \cap (B_1 \setminus B_a)$. We may assume that $H(u) > 0$ in $W \cap (B_1 \setminus B_a)$, for otherwise, replace u by $-u$. For a constant λ, consider

$$\psi_\lambda(x) = \lambda(1 - |x|), \qquad \text{for all } x \in B_1 \setminus B_a.$$

Because $u = 0$ and $|Du| = 0$ on $\partial W \cap (B_1 \setminus B_a)$, we can choose a sufficiently large λ so that

$$\psi_\lambda > u \quad \text{on } W \cap (B_1 \setminus B_a).$$

Let us consider a smaller domain $W \cap (B_1 \setminus B_{a'})$ for some $a < a' < 1$. We continuously decrease λ until, for the first time, $\psi_\lambda(x_0) = u(x_0)$ at some $x_0 \in W \cap (B_1 \setminus B_{a'})$. By Proposition 4.2, we have $u(x_0) > 0$ and $\lambda > 0$. Furthermore, we have

$$|Du| \geq |D_r u| \geq |D_r \psi_\lambda| = \lambda \quad \text{at } x_0, \tag{4.1}$$

where $D_r = \sum_i (x^i/|x|)\partial_i$.

Let $\epsilon = u(x_0)$, and consider the level set

$$\Sigma_\epsilon := \{(x, \epsilon) \in \mathbb{R}^{n+1} : x \in W \cap (B_1 \setminus B_a),\ u(x) = \epsilon\}.$$

Because $|Du|(x_0) \neq 0$, Σ_ϵ is smooth near x_0. Let $\eta = \phi Du/|Du|$. Then, η is a unit normal to Σ_ϵ in $(\{x^{n+1} = \epsilon\}, g_S)$ near x_0. Denote by H_{Σ_ϵ} the mean curvature of Σ_ϵ in $\{x^{n+1} = \epsilon\}$ with respect to η. Since $H(u) > 0$ and $R \geq n(n-1)$, by Corollary 3.4 we obtain

$$0 \leq -\frac{|Du|}{\sqrt{1+|Du|^2}} H_{\Sigma_\epsilon} + \frac{(n-1)u}{\sqrt{1+|Du|^2}} \quad \text{at } x_0.$$

By above inequality and (4.1), at x_0,

$$H_{\Sigma_\epsilon} \leq \frac{(n-1)u}{|Du|} \leq (n-1)\frac{\psi_\lambda}{|D_r\psi_\lambda|} = (n-1)(1-|x_0|). \tag{4.2}$$

On the other hand, note that $\Sigma' := S_{|x_0|} \times \{\epsilon\}$ is the level set of ψ_λ and is tangent to Σ_ϵ at x_0. By taking trace of (3.4) and computing directly, the mean curvature $H_{\Sigma'}$ of Σ' in $(\{x^{n+1} = \epsilon\}, g_S)$ with respect to the inward unit normal vector is

$$H_{\Sigma'} = \left(\frac{n-1}{|x_0|}\right) \frac{1+\epsilon^2 - |x_0|^2}{2}.$$

By the construction, we have at x_0 that

$$H_{\Sigma_\epsilon} \geq H_{\Sigma'} > (n-1)(1-|x_0|).$$

This contradicts with (4.2). $\square$

For the rest of the section, let M be a compact hypersurface in $\mathbb{S}^{n+1}$ with nonempty boundary ∂M. Denote by A, H, and R, the second fundamental form, mean curvature, and scalar curvature of M in $\mathbb{S}^{n+1}$, respectively. Let $\mathrm{int}(M)$ be the set of *interior* points in M, i.e., $\mathrm{int}(M) = M \setminus \partial M$. The set M_0 of *interior geodesic points* is given by

$$M_0 = \{p \in \mathrm{int}(M) : A(p) = 0\}. \tag{4.3}$$

The following lemma gives a useful characterization of M_0. This is the only place that requires M to be C^{n+1} rather than C^2. The proof is parallel to the Euclidean case (see, for example, [15] and [10]).

Lemma 4.4. *Suppose that $M_0 \neq \emptyset$. Let M_0' be a connected component of M_0. Then M_0' is contained in a great n-sphere $\mathbb{S}^n$ of $\mathbb{S}^{n+1}$ which is tangent to M at every point of M_0'.*

Proof. Consider the Gauss map $\nu : \mathrm{int}(M) \to \mathbb{S}^n$. Since M is of C^{n+1}, the Gauss map is of C^n. Note that the Gauss map ν has rank zero at any point of M_0. We can then apply a theorem of Sard [16, p. 888, Theorem 6.1], to obtain that image $\nu(M_0)$ has Hausdorff dimension zero in $\mathbb{S}^n$. It follows that $\nu(M_0)$ is totally disconnected in $\mathbb{S}^n$. Then $\nu(M_0')$ consists of a single point, denoted by ν_0.

It remains to show that M_0' lies in a great sphere which is orthogonal to ν_0. Let $p_0 \in M_0'$. Consider the stereographic projection from $\mathbb{S}^{n+1}$ to $\mathbb{R}^{n+1}$ so that p_0 is mapped to the origin of $\mathbb{R}^{n+1}$. Let $(V; y_1, \dots, y_n)$ be local coordinates centered at p_0 in M, and define

$$\varphi(y) = g_S(\nu_0, x(y) - x(0)), \qquad \text{for each } y = (y_1, \dots, y_n) \in V.$$

Here $x = (x_1, \dots, x_{n+1})$ is the coordinates in $\mathbb{R}^{n+1}$ such that $x(0) = 0$, and g_S is the spherical metric on $\mathbb{R}^{n+1}$. Then, the function $\varphi \in C^{n+1}(V)$, and by our construction,

$$M_0' \cap V \subset \big\{ y \in V \mid \frac{\partial \varphi}{\partial y_i}(y) = 0,\ i = 1, \dots, n \big\}.$$

It follows from a theorem of A. P. Morse [13, p. 70, Theorem 4.4] that φ is a constant on $M_0' \cap V$; thus, $\varphi \equiv 0$ on $M_0' \cap V$. Since M_0' is connected,

$$g_S(\nu_0, x(y) - x(0)) = 0 \qquad \text{for all } y \in V,$$

that is, M_0' lies in the hyperplane through the origin which is orthogonal to ν_0. Therefore, M_0' lies in a great n-sphere orthogonal to ν_0. □

Before we proceed to prove Theorem 2, let us make precise some terminology.

Definition 4.5. For a C^2 hypersurface $M \subset \mathbb{S}^{n+1}$ with boundary ∂M, we say that M *is tangent to a great n-sphere* $\mathbb{S}^n \subset \mathbb{S}^{n+1}$ *at* ∂M, if ∂M lies in $\mathbb{S}^n$, and a neighborhood of ∂M in M can be represented as the graph of a C^2 function u over an open subset V in $\mathbb{S}^n$ with $u = 0, |Du| = 0$ on ∂M. Furthermore, we say that M *is tangent to a hemisphere* $\mathbb{S}^n_+ \subset \mathbb{S}^{n+1}$ *at* ∂M, if $\partial M \subset \mathbb{S}^n_+$ and the open subset V given above lies in the component of $\mathbb{S}^n_+ \setminus \partial M$ which does not contain $\partial \mathbb{S}^n_+$. (In the case that $\partial M = \partial \mathbb{S}^n_+$, V is simply an open subset in the open hemisphere).

Remark 4.6. Equivalently, a C^2 hypersurface $M \subset \mathbb{S}^{n+1}$ is tangent to a hemisphere $\mathbb{S}^n_+ \subset \mathbb{S}^{n+1}$ at ∂M if there exists a stereographic projection from $\mathbb{S}^{n+1}$ to $\mathbb{R}^{n+1}$ such that the following properties hold:

1. (1) $\mathbb{S}^n_+$ is mapped onto the closed unit ball $\overline{B}_1$ in $\{x^{n+1} = 0\}$, and ∂M is mapped onto a closed subset in $\overline{B}_1$;
2. (2) a neighborhood of ∂M in M can be represented as the graph of a function $u \in C^2(\overline{V})$ over an open subset $V \subset \{x^{n+1} = 0\}$ such that $u = 0$ and $|Du| = 0$ on ∂M (here we identify ∂M with its image);
3. (3) V is contained in the region enclosed by ∂M.

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Lemma 4.4, it is sufficient to show that $M = \overline{M_0}$, where M_0 is given by (4.3). Since $R \geq n(n-1)$, by the Gauss equation,

$$H^2 = |A|^2 + R - n(n-1) \geq |A|^2.$$

Thus, $M_0 = \{p \in \mathrm{int}(M) : H(p) = 0\}$. We would like to show that $H \equiv 0$ on M.

First, we claim that $\overline{M_0} \cap \partial M \neq \emptyset$. Since M is tangent to a hemisphere $\mathbb{S}^n_+$ at ∂M, by Remark 4.6 there exists a stereographic projection and an open subset V in $\overline{B}_1 \subset \{x^{n+1} = 0\}$ such that a neighborhood of ∂M in M is the graph of a C^{n+1} function u over V in $(\mathbb{R}^{n+1}, g_S)$ and that $u = 0$ and $|Du| = 0$ on ∂M. Applying Lemma 4.3 with $W = V$ proves the claim.

Next, for any connected component M_0' of M_0 such that $\overline{M_0'} \cap \partial M \neq \emptyset$, by Lemma 4.4, $\overline{M_0'}$ lies entirely in $\{x^{n+1} = 0\}$. We denote by M_* the union of all such components $\overline{M_0'}$ together with ∂M. Then, M_* is a closed and connected subset in $\{x^{n+1} = 0\}$. We want to show that $M_* = M$. Suppose not. Then, there exists a bounded open set $W \subset \{x^{n+1} = 0\}$ enclosed by M_*, such that H is never zero over a neighborhood of ∂W intersecting W. Furthermore, note that $W \subset B_1$, because of Remark 4.6 (3) and the fact that M has no self-intersection. On the other hand, observe that M is tangent to $\{x^{n+1} = 0\}$ at ∂W. Thus, M near ∂W can be represented by the graph of a C^{n+1} function v over an open subset in W, and $v = 0$ and $|Dv| = 0$ on ∂W. Applying Lemma 4.3 yields a contradiction. Hence, $M = M_*$. ☐

5. Rigidity in Euclidean space

In this section, we shall prove Theorem 5. Throughout this section, we denote by M a connected, embedded, and orientable C^{n+1} hypersurface in $\mathbb{R}^{n+1}$ with nonempty boundary ∂M, unless otherwise indicated. We first recall the lemma about the Euclidean mean curvature operator that we established earlier in [10]. Let f be a C^2 function defined over an open set in $\mathbb{R}^n$. The upward unit normal vector of the graph of f in $(\mathbb{R}^{n+1}, g_0)$ is

$$\nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}.$$

The mean curvature operator is defined by

$$H(f) = -\mathrm{div}_{g_0} \nu = \sum_{i,j=1}^n \left(\delta_{ij} - \frac{f_i f_j}{1 + |Df|^2} \right) \frac{f_{ij}}{\sqrt{1 + |Df|^2}}.$$

Lemma 5.1 ([10])**.** *Let W be a bounded open subset in $\mathbb{R}^n$ and U be an open neighborhood of ∂W in $\mathbb{R}^n$. Suppose $f \in C^2(\overline{U \cap W})$ and $f = 0, |Df| = 0$ on $\partial W \cap U$. If the scalar curvature of the graph of f is nonnegative, then $H(f) = 0$ at some points in $U \cap W$.* ☐

Definition 5.2. For a C^2 hypersurface $M \subset \mathbb{R}^{n+1}$ with boundary ∂M, we say that ∂M is *planar* if the following conditions are satisfied:

(1) ∂M is contained in a hyperplane;
(2) a neighborhood of ∂M in M can be represented as the graph of a C^2 function f over a domain Ω in the hyperplane, satisfying that $f = 0$ and $|Df| = 0$ on ∂M;
(3) Ω is contained in the region enclosed by ∂M.

We recall the following characterization of the set of geodesic points of complete hypersurfaces, due to Sacksteder [15] (see also [10]). For our proof of Theorem 5, this is the only place that requires the hypersurface M to be C^{n+1}, instead of C^2. We denote the set of interior geodesic points by

$$M_0 = \{p \in \operatorname{int}(M) : A(p) = 0\}.$$

Lemma 5.3. *Let M be a C^{n+1} hypersurface in $\mathbb{R}^{n+1}$ with boundary ∂M. Let M_0 be the set of interior geodesic points defined above. Let M_0' be a connected component of M_0. Then M_0' lies in a hyperplane which is tangent to M at every point in M_0'.* □

We now prove Theorem 5.

Proof of Theorem 5. By the Gauss equation and the assumption $R \geq 0$,

$$H^2 = |A|^2 + R \geq |A|^2.$$

That is, $M_0 = \{p \in \operatorname{int}(M) : H(p) = 0\}$.

Let M_* be a connected component of $M_0 \cup \partial M$ such that $M_* \cap \partial M \neq \emptyset$. By Lemma 5.3 and the condition that ∂M is planar, $M_* \subset \{x^{n+1} = 0\}$. It suffices to prove $M_* = M$. By Definition 5.2 (3), M_* lies in the closed subset in $\{x^{n+1} = 0\}$ enclosed by ∂M. Suppose that $M_* \neq M$. Then, there exists a bounded open set $W \subset \{x^{n+1} = 0\}$ enclosed by M_*, and H is never zero over a neighborhood of ∂W intersecting with W. On the other hand, observe that M is tangent to $\{x^{n+1} = 0\}$ at M_*. Applying Lemma 5.1 yields that H must vanish at points in W which are arbitrary close to ∂W. This is a contradiction. Thus, $M_* = M$, and M is contained in $\{x^{n+1} = 0\}$. □

6. Examples

In this section, we shall present two examples to demonstrate that the boundary conditions in Theorem 2 and Theorem 5 are indispensable. Thus, the results are optimal in this sense.

Let us first consider Theorem 5, where ∂M is assumed to be *planar* (see Definition 5.2). In particular, by (3) in Definition 5.2 we require that M is tangent to ∂M from the bounded region enclosed by ∂M. This is necessary, for, in the following example, M has nonnegative scalar curvature but is tangent to its boundary ∂M from the region outside of ∂M.

Example 6.1. Consider the surface in Euclidean space $(\mathbb{R}^3, g_0)$:

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2} - f(z) = 0\},$$

where

$$f(z) = (\sqrt{z} + 1)\sqrt{1 - z^2} \qquad \text{for all } 0 \leq z \leq 1.$$

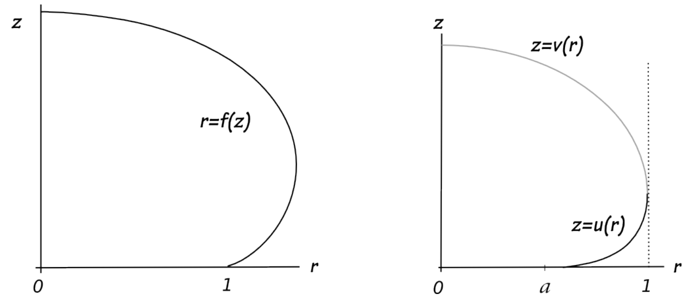

FIGURE 1. The left figure indicates the graph of the function $r = f(z)$ in Example 6.1. The right figure indicates the graphs of $z = u(r)$ and $z = v(r)$ in Example 6.2. The surfaces are obtained by rotating the curves about the z-axis.

Clearly, the surface M is obtained by rotating the curve $f(z)$ about the z-axis. Note that M has nonnegative Gauss curvature, because $f(z)$ is concave. Furthermore, M is smooth with boundary

$$\partial M = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

It is readily to check that ∂M satisfies all conditions except (3) in Definition 5.2, since M is tangent to the plane $\{z = 0\}$ at ∂M from the region outside of ∂M. $\square$

Next, let us consider Theorem 2, where M is assumed to be *tangent to a great n-hemisphere at ∂M* (see Definition 4.5 and Remark 4.6). Similarly to the Euclidean case, we remark that condition (3) in Remark 4.6 cannot be removed, in view of the following example.

Example 6.2. Consider $\mathbb{R}^3$ with the spherical metric g_S, i.e.,

$$g_S = \frac{4g_0}{(1 + x^2 + y^2 + z^2)^2},$$

where g_0 is the Euclidean metric and (x, y, z) are the Cartesian coordinates. We denote $r = \sqrt{x^2 + y^2}$. Let $0 < a < 1$ be a constant. Define

$$u(r) = \frac{1}{\sqrt{2}} \left(-2\sqrt{1 - r} - \frac{r}{\sqrt{1 - a}} + 2\sqrt{1 - a} + \frac{a}{\sqrt{1 - a}} \right),$$

for all $a \le r \le 1$; and define

$$v(r) = u(1) + \sqrt{1 - r^2} = \sqrt{\frac{1 - a}{2}} + \sqrt{1 - r^2}, \qquad \text{for all } 0 \le r \le 1.$$

Let M be the surface which is the union of graph of u over $[a, 1]$ and the graph of v over $[0, 1]$. Then, we *claim* that M is a C^2 surface of scalar curvature $R \ge 2$ in $(\mathbb{R}^3, g_S)$ (where $R = 2$ holds at and only at the boundary points). It is straightforward to check that

$$\partial M = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2 < 1\}$$

satisfies all the conditions in Remark 4.6 except (3).

It suffices to show the claim. Note that the graph of v is a portion of the unit 2-sphere centered at $(r, z) = (0, \sqrt{(1-a)/2})$. With respect to the upward unit normal, the graph of v has principal curvatures

$$\kappa_1 = \kappa_2 = -\frac{[u(1)]^2}{2} = -\frac{1-a}{4}.$$

Hence, $R = 2 + 2\kappa_1\kappa_2 > 2$ on the graph of v. For the function u, obviously we have $u \in C^\infty([a, 1)) \cap C^0([a, 1])$. In addition, u satisfies the following properties:

$$u(a) = 0, \quad u'(a) = 0, \quad u'(r) > 0 \quad \text{for all } a < r < 1, \quad \text{and} \tag{6.1}$$

$$u''(r) > u'[1 + (u')^2], \qquad \text{for all } a \le r < 1. \tag{6.2}$$

With respect to the upward unit normal, the principle curvatures of the graph of u are given by

$$\begin{aligned}
\lambda_1 &= \frac{1}{\sqrt{1+(u')^2}}\left[u - ru' + \frac{1+u^2+r^2}{2}\frac{u''}{1+(u')^2}\right]\\
\lambda_2 &= \frac{1}{\sqrt{1+(u')^2}}\left[u - ru' + \frac{1+u^2+r^2}{2}\frac{u'}{r}\right]\\
&= \frac{1}{\sqrt{1+(u')^2}}\left[u + u'\frac{1+u^2-r^2}{2}\right].
\end{aligned}$$

Since $r \le 1$, by (6.1) we have $\lambda_2 \ge 0$. Applying (6.2) to λ_1 yields that

$$\begin{aligned}
&u - ru' + \frac{1+u^2+r^2}{2}\frac{u''}{1+(u')^2}\\
&> u - ru' + \frac{1+u^2+r^2}{2}u' = u + u'\frac{u^2+(r-1)^2}{2} \ge 0.
\end{aligned}$$

It follows that $\lambda_1 > 0$. Therefore, $R = 2 + 2\lambda_1\lambda_2 \ge 2$ on the graph of u, where $R = 2$ if and only if $r = a$. Obviously M is smooth at the interior points of graph u and graph v. It is elementary to verify that M is of C^2 at the intersection curve $(r, z) = (1, \sqrt{(1-a)/2})$ of the two graphs. Thus, the claim is proved.

Now, because $R > 2$ in a neighborhood of the intersection curve, we can perturb M locally near the intersection curve to get a smooth surface $\widetilde{M}$ in $(\mathbb{R}^3, g_S)$ with $R \ge 2$. Also, $\widetilde{M}$ is identical to M away from a neighborhood of the intersection curve. □

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